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The non-abelian Born-Infeld action at order F^6

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Abstract

To gain insight into the non-abelian Born-Infeld (NBI) action, we study coinciding D-branes wrapped on tori, and turn on magnetic fields on their worldvolume. We then compare predictions for the spectrum of open strings stretching between these D-branes, from perturbative string theory and from the effective NBI action. Under some plausible assumptions, we find corrections to the Str-prescription for the NBI action at order F^6. In the process we give a way to classify terms in the NBI action that can be written in terms of field strengths only, in terms of permutation group theory.

1   Introduction

Consider a flat D-brane in type II string theory. The bosonic massless degrees of freedom of an open string ending on the D-brane are a $U(1)$ gauge field, associated to excitations of the string longitudinal to the brane, and neutral scalar fields, associated to transverse excitations of the brane. The effective action for these massless degrees of freedom

for slowly varying field strengths is known up to all orders in the string length $\sqrt{\alpha'}$. It is the Born-Infeld action [2]:

$$\mathcal{S} = -T_p \int d^{p+1}\sigma \sqrt{\det(\delta_\alpha{}^\beta + 2\pi\alpha' F_\alpha{}^\beta)}. \tag{1}$$

Expanding this action in the field strength, we obtain a Maxwell action with higher order corrections in $\alpha' F$.

When N D-branes coincide, the massless degrees of freedom of open strings beginning and ending on them are a $U(N)$ gauge field, and a number of scalar fields in the adjoint of the gauge group. The extra degrees of freedom come from strings stretching from one D-brane to another that become massless when these D-branes coincide. A problem that seems to appear naturally by analogy with the abelian case, is to write down the effective action for these massless degrees of freedom, for slowly varying field strengths. In fact, we know the first terms of such a non-abelian Born-Infeld (NBI) action exactly. From string scattering amplitudes [1] and a three-loop betafunction calculation [2], we know that the expansion of the NBI lagrangian in powers of the field strengths begins with [3]:

$$\begin{aligned}
\mathcal{L} = {} & Tr(\frac{1}{4}F^{\alpha_1\alpha_2}F_{\alpha_2\alpha_1} \\
& + \frac{1}{24}F^{\alpha_1\alpha_2}F_{\alpha_2\alpha_3}F^{\alpha_3\alpha_4}F_{\alpha_4\alpha_1} + \frac{1}{12}F^{\alpha_1\alpha_2}F^{\alpha_3\alpha_4}F_{\alpha_2\alpha_3}F_{\alpha_4\alpha_1} \\
& - \frac{1}{48}F^{\alpha_1\alpha_2}F_{\alpha_2\alpha_1}F^{\beta_1\beta_2}F_{\beta_2\beta_1} - \frac{1}{96}F^{\alpha_1\alpha_2}F^{\beta_1\beta_2}F_{\alpha_2\alpha_1}F_{\beta_2\beta_1} \\
& + \mathcal{O}(F^6)).
\end{aligned} \tag{2}$$

Through this order, this coincides with the expansion of the symmetrized trace action [3]:

$$\mathcal{S} = -\int d^{p+1}\sigma \, Str(\det\sqrt{\delta_\alpha{}^\beta + F_\alpha{}^\beta}), \tag{3}$$

where the prescription is to formally expand the square root and the determinant in F first, then to symmetrize over all orderings of the field strength factors, and finally to perform the trace.

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[2]The Dp-brane tension we denote T_p, and $\alpha \in \{0, 1, \ldots, p\}$. We choose the static gauge and we leave out the transverse scalars for reasons to be explained below.

[3]We put $2\pi\alpha' = 1$ from now on, ignore the overall factor T_p, and an additive constant.

There is some ambiguity in the expression of the NBI action in terms of field strengths and their covariant derivatives, since $[D_\alpha, D_\beta]F_{\gamma\delta} = i[F_{\alpha\beta}, F_{\gamma\delta}]$. One could rewrite expression (2) by assembling the second with the third term (and the fourth with the fifth) at the cost of introducing extra $[D, D]FFF$ terms. The all order proof [3] of the symmetric trace formula is only claimed to be valid up to this type of terms, and therefore pertains only to the sum of the coefficients of the second and third terms (and likewise for the fourth and fifth), which in fact also follows from specialising to the abelian case. Nevertheless it is remarkable that, to fourth order, the symmetric trace gives the complete expression for the superstring (though not for the bosonic string), and thus it deserves to be investigated in detail at higher orders. In this paper, we will embed the symmetric trace hypothesis into a more general action. Since we are approximating the NBI at string tree level, we do keep the restriction of considering only an overall trace in the fundamental over the gauge group factors. Expanding for 'slowly varying field strengths' is admittedly ambiguous, and an unambiguous order would add the number of F's to twice the number of D's. We will not include the most general possibility, but limit ourselves to a subset adapted to the exploratory program that we propose in the next section: all terms where the covariant derivatives occur in antisymmetric combinations, and can therefore be written purely in terms of the field strengths, are included in our analysis, but symmetric derivative combinations are not. In other words, we adopt here the definition that acceleration terms are expressed as symmetrized products of covariant derivatives.

A direct calculation of the F^6 terms would imply the study of a 6-gluon open string amplitude or a 5-loop β-function. Both are technically very involved. In the next we will develop a simpler approach which will allow us to determine the F^6 term to a large extent.

2   Wrapped D-branes and the NBI action

Magnetic field strengths on tori

In this section, we map out our testing ground for any proposal for the NBI action. Consider N coinciding D2n-branes, wrapped around a T^{2n} torus. Switch on constant magnetic fields in the Cartan subalgebra (CSA) of the $U(N)$ gauge group. These correspond to embedded D-branes of

lower dimension. Choose the magnetic fields to be blockdiagonal in the Lorentz indices, for simplicity. The plan [4] is now to compare the spectrum for small fluctuations around this background as predicted by string theory, with the spectrum predicted by the proposed non-abelian Born-Infeld. Since we only want to consider the (originally) massless degrees of freedom of the open string, we decouple the massive modes by sending $\alpha' \to 0$. To maintain the relevance of the non-linear corrections to Yang-Mills theory prescribed by the NBI, we crank up the magnetic field to keep $\alpha' F$ constant.

Perturbative string theory spectrum

To write down the spectrum for the low-lying modes predicted by perturbative string theory, we need some notation. Suppose we restrict to the situation in which we have only 2 D2n-branes [4]. Since the magnetic background is in a CSA, we can diagonalize it, and associate a magnetic field strength to each of the two branes, $\mathcal{F}^{(1)}_{2i-1,2i}$ and $\mathcal{F}^{(2)}_{2i-1,2i}$. We chose the background to be blockdiagonal in the Lorentz indices. T-dualizing along the 2, 4, ..., $2n$ directions, we end up with two Dn-branes at angles given by:

$$\tan \gamma_i^{(n)} = \mathcal{F}^{(n)}_{2i-1,2i}. \tag{4}$$

Then the modes of the open string connecting the two Dn-branes, which correspond to the off-diagonal gauge field modes in the directions $2k - 1, 2k, \ k \in \{1, \cdots, n\}$, have a spectrum [5]:

$$M_k^2 \;=\; \sum_{i=1}^{n} (2m_i + 1)\epsilon_i \mp 2\epsilon_k, \tag{5}$$

$$\epsilon_i \;=\; \gamma_i^{(1)} - \gamma_i^{(2)}. \tag{6}$$

The details of how to compute this spectrum can be found in [7] and some handy formulas are in [5].

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[4]We will do this throughout this paper. The rationale is that perturbative string theory as well as the linear analysis we perform is only sensitive to the interactions between each pair of D-branes [5].

[5]The modes of the scalar fields and the fermions have a similar spectrum, and we do not expect them to provide any additional information [6] [5].

4

Yang-Mills analysis

To gain some intuition for how this spectrum comes about and to prepare for the treatment in the case of the effective action, we take a look at the Yang-Mills approximation to the problem. Consider then the Yang-Mills truncation of the non-abelian Born-Infeld action. We can study the same background as before, and determine the spectrum of the fluctuations around the background in this approximation. This was done in full detail in [8] [6]. The result is:

$$ M_k^2 = \sum_{i=1}^{n} \left\{ (2m_i + 1)(\mathcal{F}^{(1)}_{2i-1,2i} - \mathcal{F}^{(2)}_{2i-1,2i}) \right\} \mp 2(\mathcal{F}^{(1)}_{2k-1,2k} - \mathcal{F}^{(2)}_{2k-1,2k}) \quad (7) $$

where for convenience, we chose $\mathcal{F}^{(1)}_{2i-1,2i} > \mathcal{F}^{(2)}_{2i-1,2i}$. It is clear that for small field strengths the string spectrum (5) reduces to the Yang-Mills spectrum (7), as expected.

The Yang-Mills spectrum can be argued for as follows. An endpoint of a string ending on one of these D-branes behaves as an electric charge in a magnetic field. The corresponding Landau problem has a harmonic oscillator spectrum with frequency proportional to the magnetic field. The other endpoint of the string acts as a particle with the opposite charge. This makes intuitive the fact that for the global motion of the string, the difference between the field strengths on the two branes acts as spacing of the energy levels. The zero-point energy, moreover, can be attributed to a Zeeman splitting of the energy levels due to the fact that different combinations of the gauge field in directions $2k-1, 2k$ have spin ± 1 under the $SO(2)$ associated to these directions.

String theory as rescaled YM

String theory adds a non-linearity to this spectrum that can for instance be intuitively understood in the T-dual picture, where magnetic fields are interchanged for rotated branes. (See [4] and [5] for instance.) For our purposes, the important observation is that the string spectrum is merely a rescaled Yang-Mills spectrum. Denoting

$$ f_i^0 = \frac{1}{2}(\mathcal{F}^{(1)}_{2i-1,2i} + \mathcal{F}^{(2)}_{2i-1,2i}), \tag{8} $$

$$ f_i^3 = \frac{1}{2}(\mathcal{F}^{(1)}_{2i-1,2i} - \mathcal{F}^{(2)}_{2i-1,2i}), \tag{9} $$

the spectrum is rescaled by a factor

$$\alpha_i^2 \equiv \frac{\epsilon_i}{2f_i^3} \quad = \quad \frac{\arctan\left(\frac{2f_i^3}{1+(f_i^0)^2-(f_i^3)^2}\right)}{2f_i^3} \tag{10}$$

for field strength fluctuations in directions $2i-1, 2i$.

A clearcut question is then, whether a proposal for the NBI action reproduces this rescaled Yang-Mills spectrum predicted by perturbative string theory. This was investigated in detail for the Str-prescription in [5] (expanding on the initial explorations in [4] and [9]). For the simplest case, on T^2, the symmetrized trace prescription yielded a spectrum with the same structure as the Yang-Mills spectrum, but with incorrect spacings. The disagreement shows up from third order on, confirming the veracity of the F^2 and F^4 terms. This clearly demonstrates that the Str-prescription is too crude an approximation to the NBI to yield the correct mass spectrum on our testing ground. For T^4, the situation remained unclear since the complete spectrum predicted by the Str-action remained undetermined. For BPS configurations on T^4, the Str-action reproduces precisely the right spectrum, but for other settings, it seems highly unlikely that the Str-prescription would lead to the correct results. On T^6, the Str would probably not yield the right spectrum even for BPS configurations [5].

The spectrum as predicted by string theory is a *rescaled* Yang-Mills *spectrum,* compare equations (5) and (7). Therefore, we will assume that the action relevant for this physical situation, should yield a *rescaled* Yang-Mills *action* for the fluctuations, meaning that it can be brought back to a Yang-Mills action by a suitable coordinate transformation. This is certainly the simplest and perhaps the most natural way to reproduce the desired string theory results. In these circumstances it seems less natural to allow in the Lagrangian terms containing derivatives that cannot be written as combinations of field strengths. Not in the least, they would make it much more difficult in practice to obtain results for the spectrum, since one would be trying to diagonalize higher order operators. As indicated before, to obtain this rescaled Yang-Mills action we do include terms to the action corresponding to all possible orderings and Lorentz contractions of field strengths. There might be an a posteriori justification for this approach, if one could prove that for the fluctuation eigenfunctions – they can explicitly be written down in terms of theta-functions as in [6] – other kinds of derivative terms are suppressed.

From the formula for the rescaling factor, we expect only terms in the lagrangian with an even number of field strengths to contribute in

our backgrounds. For this reason, we do not consider terms with an odd number of field strengths.

BPS conditions

As already pointed out in [5], the translation of the BPS conditions in string theory in terms of the background field strength in the effective action might provide an additional handle on the NBI action. Concretely, in section 6 we will investigate what constraints on the NBI follow from the demand that self-dual configurations on T^4 should solve the equations of motion.

These constraints on the action are a priori independent from the ones obtained from the analysis along the line discussed in the previous subsection. They turn out to provide an independent check on some of the results obtained with the rescaled YM program, and also to give additional constraints on the NBI action.

3   The NBI at order F^4

We start by carrying out the program proposed above at the first non-trivial level, the F^4 terms in the non-abelian Born-Infeld. This will serve to illustrate the method we use in a simple setting. Moreover, it will turn out that the straightforward spectral analysis, under the assumptions we make, is able to replace a four point function computation in open string theory, or a three-loop beta-function computation in a non-linear σ-model approach, demonstrating the power of our method.

The most general lagrangian we can write down under the stated restrictions (see page 3, 3, 6) is then:

$$
\begin{aligned}
\mathcal{L} \;=\; & Tr(\\
& a_1^2 F_{\alpha_1\alpha_2} F^{\alpha_2\alpha_1} \\
& + a_1^4 F_{\alpha_1\alpha_2} F^{\alpha_2\alpha_3} F_{\alpha_3\alpha_4} F^{\alpha_4\alpha_1} + a_2^4 F_{\alpha_1\alpha_2} F_{\alpha_3\alpha_4} F^{\alpha_2\alpha_3} F^{\alpha_4\alpha_1} \\
& + a_1^{2,2} F_{\alpha_1\alpha_2} F^{\alpha_2\alpha_1} F_{\beta_1\beta_2} F^{\beta_2\beta_1} + a_2^{2,2} F_{\alpha_1\alpha_2} F_{\beta_1\beta_2} F^{\alpha_2\alpha_1} F^{\beta_2\beta_1}) .
\end{aligned}
$$

$$(11)$$

At this low order, it is easy to check that these are indeed the only linearly independent terms. At higher order the analysis becomes untransparant. In section 4, we will therefore introduce a diagrammatic representation for these terms.

The symmetric trace prescription would relate the coefficients in equation (11) by $a_2^4 = 2a_1^4 = -4a_1^{2,2} = -8a_2^{2,2}$ and the determinant formula sets this equal to $\frac{a_1^2}{3}$. Let us see how this result comes about by imposing the correspondence of the spectrum with equation (7) with the rescaling factor (10). First of all, we demand that the abelian action be reproduced if we restrict to a $U(1)$ subgroup. The 3 constraints this yields on the coeficients are easy to determine and they are listed in appendix C equation (39). Next, we determine the action quadratic in off-diagonal gauge field fluctuations, in a background blockdiagonal in the Lorentz indices. We restrict to a $U(2)$ subgroup since we always work with 2 branes only. The action for the quadratic fluctuations in this background is given in appendix D equations (48) and (49), for second and fourth order respectively. Its structure is as follows:

$$
\begin{aligned}
\mathcal{L}^{(2,4)} \;=\;& c_i^{\text{kin}}(f,a)\left((\delta_1 F_{0,2i-1}^{(a)})^2 + (\delta_1 F_{0,2i}^{(a)})^2\right) \\
& -c_i(f,a)(\delta_1 F_{2i-1,2i}^{(a)})^2 \\
& -\frac{1}{2}c_{ij}(f,a)\sum_{i\neq j}\left((\delta_1 F_{2i-1,2j-1}^{(a)})^2 + (\delta_1 F_{2i-1,2j}^{(a)})^2 + \right. \\
& \qquad\qquad\qquad \left. \delta_1 F_{2i,2j-1}^{(a)})^2 + (\delta_1 F_{2i,2j}^{(a)})^2\right) \\
& +c_{ij}^{\text{nym}}(f,a)(\delta_1 F_{2i-1,2i}^{(a)}\delta_1 F_{2j-1,2j}^{(a)}) \\
& -2c_i^{\text{quad}}(f,a)\delta_2 F_{2i-1,2i}^{(3)}f_i^3\,,
\end{aligned}
\tag{12}
$$

where (see appendix D) in $c(.,.)$-coefficients f represent the background field strength values, a stands for the coefficients a_n^k of equation (11),

$$
\begin{aligned}
\delta_1 F_{\alpha\beta} &= D_\alpha \delta A_\beta - D_\beta \delta A_\alpha\,, & (13) \\
\delta_2 F_{\alpha\beta} &= i[\delta A_\alpha, \delta A_\beta]\,, & (14) \\
D_\alpha &= \partial_\alpha + i[A_\alpha, .]\,, & (15)
\end{aligned}
$$

and the superscript (a) runs over two orthogonal non-CSA $SU(2)$ components.

The different lines are treated as follows:

- The first line is the kinetic term. The first step in the comparison with the Yang-Mills action is a rescaling of the fluctuations of the gauge potentials such that the kinetic term has the standard normalisation:

$$
\begin{aligned}
\delta A_n &= b_i^{-1}\delta a_n \text{ for } n \in \{2i-1, 2i\}\,, & (16) \\
c_i^{\text{kin}} &= b_i^2\,. & (17)
\end{aligned}
$$

8

- The second line represents the deformation energy of the modes in directions $2i - 1, 2i$. By a rescaling of the space coordinates,

$$X_n = b_i \gamma_i x_n \text{ for } n \in \{2i - 1, 2i\} , \quad (18)$$
$$c_i = \gamma_i^2 b_i^4 , \quad (19)$$

it is brought to the standard Yang-Mills form with a rescaled background potential

$$a_n = b_i \gamma_i A_n \text{ for } n \in \{2i - 1, 2i\} . \quad (20)$$

- In the third line, the rescalings above destroy the Yang-Mills structure unless, when $c_{ij} \neq 0$, we have that $\gamma_i = \gamma_j (= \gamma)$. This being granted, the overall factor agrees with the Yang-Mills value provided

$$c_{ij} = b_i^2 b_j^2 \gamma^2 . \quad (21)$$

- The fourth line is absent from the Yang-Mills action. In accordance with our assumptions we put their coefficients c_{ij}^{nym} to zero.

- The fifth line contains the terms linear in the second order fluctuation $\delta_2 F_{\mu\nu} = [\delta A_\mu, \delta A_\nu]$. They have to follow the same scaling as the second line, but in fact this is not an independent condition. If the Yang-Mills structure of the third line is imposed, this follows from the fact that fluctuations of the background configuration that are gauge transformations leave the action unchanged.

- Additional terms arise, with the structure $\delta_1 F_{2j,2i}^{(a)} \delta_1 F_{2i-1,2j-1}^{(a)} - \delta_1 F_{2j-1,2i}^{(a)} \delta_1 F_{2i-1,2j}^{(a)}$. Partial integration can be combined with the Lie-algebraic structure of these terms to absorb them into the second, fourth and fifth lines.

Summarising: the non-Yang-Mills terms have to be put to zero, and then $c_i (c_i^{\text{kin}})^{-2} = \gamma^2$ should be independent of i, and $c_i (c_i^{\text{kin}})^{-1}$ should equal the required scaling factor α_i^2 of equation (10). These demands uniquely fix (after normalizing $a_1^2 = 1/4$) all coefficients in the action (11):

$$a_1^4 = \frac{1}{24} , \quad (22)$$
$$a_2^4 = \frac{1}{12} , \quad (23)$$
$$a_1^{2,2} = -\frac{1}{48} , \quad (24)$$
$$a_2^{2,2} = -\frac{1}{96} , \quad (25)$$

9

which matches the action, eq. (2), predicted by the computation of scattering amplitudes, a betafunction calculation, and the symmetric trace prescription.

4 Group theory and contractions

Diagrams

The implementation of our program at order 4 in the previous section starts from the most general action consisting of terms that could be written in terms of field strengths alone. This action is easy to write down in low orders, but at higher order, a more systematic approach is called for. In this section we will describe an attempt to bring some systematics into the classification of the different terms at order 2,4, and 6 by using permutation group theory. The explicit examples will be taken from order 6, and we will also give results at order 4, but the scheme carries over to all orders.

Let us consider some typical terms in the action at order 6:

$$Tr(F_{\alpha_1\alpha_2}F^{\alpha_2\alpha_3}F_{\alpha_3\alpha_4}F^{\alpha_4\alpha_1}F_{\beta_1\beta_2}F^{\beta_2\beta_1}) \text{ and} \tag{26}$$

$$Tr(F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}F_{\beta_1\beta_2}F^{\alpha_2\alpha_3}F^{\alpha_4\alpha_1}F^{\beta_2\beta_1}) \ . \tag{27}$$

The interplay between the Lorentz index contractions and the group theory trace can be encoded in different ways. A pictorial way is to associate a diagram to each such term, by drawing points on the corners of a regular hexagon, indicating the position of the F-factors in the trace, and lines (with arrows, which will however soon be dropped) connecting the different points, indicating the Lorentz contractions. The terms given in equations (26-27) are then represented by figure 1, where the left most F in the trace is represented by the upper left corner of the hexagon.

An alternative description, geared towards the permutation group considerations that follow, goes as follows. Label the first index on the 6 field strengths from 1 to 6. Then the sequence of indices in the second position is a permutation of the first index. We will denote this permutation $i(.)$, and use it to label the diagram. The permutations corresponding to (26-27) are (1234)(56) and (1425)(36) in a cycle notation[6]. Obviously, each term at order 6 can be represented by one (or more) of the $6! = 720$ possible permutations.

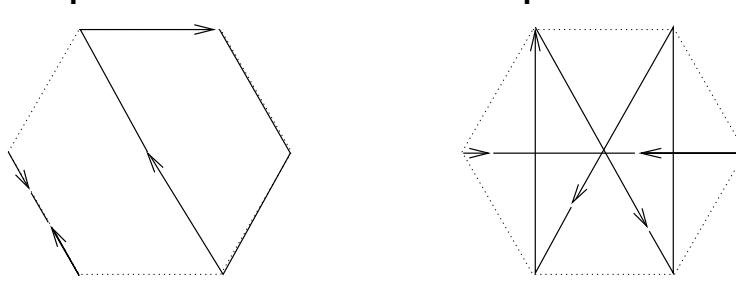[6] i.e., for the second example, $i(5) = 1, i(1) = 4, i(4) = 2$ etc.

Figure 1: Diagrammatic way of representing the terms in eqs. (26-27).

Conjugacy classes

The complex linear combinations of diagrams are taken as a representation space for the permutation group of 6 elements. The action of S_6 on this representation space is *by conjugation*, as we now explain. The action of the permutation group consists of reshuffling the vertices of the diagrams, which is the same as reshuffling F's in the trace. The action of a permutation g on the vertices becomes, after relabeling:

$$
\begin{aligned}
g(F_{1,i(1)} \ldots F_{6,i(6)}) &\equiv F_{g^{-1}(1),i(g^{-1}(1))} \ldots F_{g^{-1}(6),i(g^{-1}(6))} & (28) \\
&= F_{1,gig^{-1}(1)} \ldots F_{6,gig^{-1}(6)} \, . & (29)
\end{aligned}
$$

Evidently, the set of diagrams within one conjugation class is invariant under this action. As far as this representation of the permutation group on the diagrams is concerned, we can study each conjugation class separately. Each of these representations separately is in fact a (transitive) representation by permutation of diagrams.

The arrows on the diagrams can be dropped. Two diagrams that are the same up to the orientation of a loop are equivalent [7] since they correspond to the same term in the action (up to an unimportant sign): reversing the arrow in a loop amounts to flipping the order of the indices in all the field strengths connected by that loop.

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[7]One can easily see that this equivalence relation is compatible with the action of the group.

An induced representation

Now we analyse the representation of the permutation group on each conjugation class. Consider a specific conjugation class, choose a diagram (without the arrows) and label it i_1. The chosen diagram is invariant under a subgroup of the permutation group (acting by conjugation as above). The invariance group of i_1 we call H_1. For both our examples (26-27) the invariance group is isomorphic with $\mathbb{Z}_4 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

It is clear that every other diagram i in the conjugacy class can be reached by the action of some group element g, namely $i = g i_1 g^{-1}$. Every gh with $h \in H_1$ yields that same diagram i. Therefore the set of diagrams within a conjugacy class is the same as the set of the left cosets with respect to the invariance group (of a diagram in that conjugacy class). The action of the group on this set of cosets is the left regular action. This representation is the representation *induced* [10] by the trivial representation of H_1 on S_6. Via Fröbenius' character formula we can then decompose this induced representation in irreducible ones, using the character table of S_6. This decomposition provides an inroad into the structure of the terms in the NBI, at order F^6 and potentially beyond. Note that if we had picked a different diagam i_2 in the same conjugacy class to start with, we would have $i_2 = g i_1 g^{-1}$ for some $g \in S_6$. The invariance group $H_2 = g H_1 g^{-1}$ would yield an equivalent construction to the previous one. Therefore, $H_1 \cong H^{cc}$ is uniquely associated to a conjugacy class (c.c.). The results for the invariance groups are summarized in table 1 [8], for the relevant conjugacy classes[9]. The split into irreducible representations is assembled in table 2.

Conjugacy class	Invariance group H^{cc}
[6]	$Z_6 \otimes Z_2$
[4 2]	$Z_2 \otimes Z_4 \otimes Z_2$
[3 3]	$(Z_3)^2 \otimes S_2 \otimes (Z_2)^2$
[2 2 2]	$(Z_2)^3 \otimes S_3$

Table 1: Invariance groups associated to conjugacy classes

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[8]The symbol $\otimes$ in table 1 does not denote a direct product. It is easy to deduce from the context how the product of subgroups should be taken. The subgroups are ordered as follows. First the cyclic permutation within a loop, then the permutation of loops of equal length, finally the orientation reversal. For loops of length 2, this last group is trivial.

[9]We momentarily explain why other conjugacy classes are irrelevant.

Cyclicity and double cosets

At this stage a term in the Lagrangian corresponds to several diagrams, since the trace is cyclic: a cyclic permutation corresponds to a rotation of the diagrams. We denote the subgroup of S_6 corresponding to these rotations as $N = Z_6^c$ (where c stands for cyclicity). Then it should be clear that the cosets gH within the double coset NgH correspond to equivalent diagrams. We finally obtain therefore, that inequivalent diagrams correspond to double cosets NgH.

To count these double cosets in the left regular representation on the H-cosets, it is sufficient to count Z_6^c invariants within each irreducible component of the representation. To do that, we can use Fröbenius reciprocity and the character tables for S_6 and Z_6^c.

The results then at order 6 are the following. Conjugacy classes of S_6 with a cycle of length 1, we do not consider since a field strength contracted with itself yields a term equal to zero in the action. We have only four conjugacy classes left then. The number of double cosets in each of these conjugacy classes is summarized in table 2, along with the decomposition into irreducible representations. The diagrams corresponding to

Conjugacy class	Irreducible reps	# invariants
[6]	$60 = [6] + [21^4] + 2.[2^3] + 2.[42] + [31^3] + [321]$	14
[42]	$45 = [6] + [51] + [2^3] + 2.[42] + [321]$	9
[33]	$10 = [6] + [42]$	3
[222]	$15 = [6] + [2^3] + [42]$	5

Table 2: Irreducible components and double cosets

these invariants are drawn and labelled in appendix A.

As we already indicated, this analysis generalizes to any order and gives therefore a systematic way to count the number of unknown coefficients in the NBI action including terms written using field strengths only, at any given order.

Invariant linear combinations

In the previous analysis, we split the representation space into conjugacy classes, next into inequivalent irreducible representations, and we determined the number of double cosets within an irreducible representation. Now we would like to write down explicitly these Z_6^c invariants in terms

of the diagrams, which translates directly into terms in the action at order 6.

For most of the irreducible representations, the number of corresponding invariants is larger than one. Lacking a criterium to decide which linear combinations are most suitable, we made the following arbitrary choice. Corresponding to a specific irreducible representation of the permutation group, there is a Young diagram and a Young symmetrizer: acting with this Young symmetrizer on a specific diagram yields automatically a vector in that irrep. The resulting vector can then simply be symmetrised with respect to the Z_6^c cyclic group. The result of this procedure is one of the sought after invariants. We have recorded in table 5 (in the appendix B) a complete[10] set of combinations obtained in this way, together with the S_6 irrep in which they are found. Each line involves a choice of starting diagram, for which we found no good criterium (like an a priori guarantee to give a linearly independent combination).

Alternatively, one may project on a generically reducible subspace formed by the sum of equivalent S_6 representations using the minimal projection operator [10] $e(\mathcal{F})$ associated to a specific irrep $\mathcal{F}$. For example in the [6] class the projection on the [42] representations yields a reducible representation, $[42] \oplus [42]$. Each of these contains two Z_6^c invariants. Acting with $e(\mathcal{F})$ on a few (arbitrarily chosen) diagrams yields vectors from this reducible space, and it is easy to pick out specific Z_6^c invariants. Since this seems to offer no particular advantages (each choice of basis for the resulting invariants seems arbitrary), we do not dwell on this further.

We pause here a moment to return to the results obtained in section 3. If we carry out the group theory analysis described in the previous paragraphs at order 4, we obtain the results in table 3.

In the part of the NBI action purely in terms of field strengths at order four only two of the four potential fourth–order invariants are actually present:

$$\mathcal{L}_{NBI-F^4} = \frac{1}{4}I_2^2 + \frac{1}{24}(I_1^4 - \frac{1}{4}I_1^{22}). \tag{30}$$

The group theory that we introduced will similarly simplify the form of the NBI action at order 6. At this stage we performed only the first step, providing a catalogue of combinations that are in the different irreps, as

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[10]We have not bothered to include the results of this analysis for the terms with Lorentz contraction structure [3 3]. The reason is that, for the backgrounds we have studied, these terms give *no* contribution to the quadratic action for the fluctuation, and therefore these terms remain completely arbitrary.

Conjugacy class	irrep	Linear combination	Name
[4]	[4]	$i_1 + 2i_2$	I_1^4
[4]	[2 2]	$i_1 - i_2$	I_2^4
[22]	[4]	$i_3 + 2i_4$	I_1^{22}
[22]	[2 2]	$i_3 - i_4$	I_2^{22}

Table 3: Combinations of diagrams based on the permutation group: order 4. The square diagram is represented by i_1, the diabolo is i_2, the cross is i_3 and i_4 the two parallel lines.

recorded in table 5 in the appendix. We now proceed to impose the data from the known string spectra.

5   A NBI at order 6

Reality   A first, fairly trivial constraint on the action comes from the demand that the action be real. The complex conjugate of a term represented by a diagram, is given by the term corresponding to the mirror diagram. This can easily be seen using the hermiticity of the Lie algebra generators. We conclude that diagrams that are mirror to each other have complex conjugate coefficients. The diagrams that are mirrorsymmetric have real coefficients [11].

Note that all diagrams at order 4 were mirrorsymmetric, and therefore they all necessarily had real coefficients. This is not true at sixth order. However, it turns out that, apart from the general structure as described for the fourth order calculation in equation (12), an additional term is present at sixth order, that is off-diagonal in the $SU(2)$ components of the field fluctuations. The rescaled Yang–Mills requirement puts this to zero. This annihilates the imaginary parts of the complex conjugate coefficients so that, as a conclusion, also at sixth order *all* coefficients are real.

String spectrum data   In section 3 we executed our program of demanding a rescaled Yang-Mills action for the action quadratic in the fluctuations on our testing ground. It was succesful there in determining the coefficients of the NBI at order 4 that we know to be correct. In this

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[11]This mirror-operation is the only group operation represented on all double cosets.

section, as discussed previously, we explore which constraints are found on the NBI if we extend this analysis to order 6.

The action for the quadratic fluctuations at order 6 has virtually the same structure as that discussed in detail for order 4 in section 3. We follow the same route and rescale the action by c_i^{kin} and demand that the action is a rescaled YM action with appropriate rescaling factor. The constraints from gauge invariance (see section 3) were not imposed a priori, but were used as a check on the computation. The result is a large set of linear equations for the coefficients a_m^{nnn} of the different terms in the action (see eq. 37). Of these, 21 are independent, leaving 10 out of 31 (see table 2) of the coefficients in the general sixth order action undetermined.

Of these 10 undetermined coefficients, 3 are the coefficients of the invariants in class [3 3]: for the background we consider these invariants give vanishing contribution as we now argue[12]. The background (see section 2) has block-diagonal fieldstrengths, and therefore the Lorentz contraction of three background fields is frustrated and vanishes[13]. Consequently, the quadratic variations could only arise when the Lorentz contraction structure is $(\mathcal{F}\mathcal{F}\delta_1 F)$ times $(\mathcal{F}\mathcal{F}\delta_1 F)$. But this also vanishes, since the $k-$ sum in $\mathcal{F}_k^i \mathcal{F}_j^k$ will contain only one term, and is hence diagonal in ij. We ignore this terms in the sequel, and continue with the remaining 28 terms of table 5 in appendix B, 7 combinations out of 28 having arbitrary coefficients.

To present the result in detail, we make a change of basis. We still base our choice of combinations of diagrams on the permutation group considerations of section 4. We remind the reader that in many cases, a given irrep occurs more than once, and in addition a given irrep usually contains two invariants. For such cases, the choice of basis for specific invariants is a priori quite arbitrary, and what was written in table 5 is a 'raw' choice. With hindsight, this choice can be improved, and the result is recorded in table 4. The following changes were made:

- If the value of coefficients for a given representation is completely fixed[14], this combination was chosen as one of the basis vectors. The other basis vectors (which therefore have zero coefficients) were taken to be orthogonal with the natural metric for the diagrams.

---

[12] It is obvious that this argument extends to very many higher order terms with a structure that factorises with Lorentz contractions of an odd number of field strengths.

[13] The same argument eliminates terms arising from $\delta_2 F$, see equation 43.

[14] We are here taking into account the requirements from reality and the rescaled Yang–Mills ansatz, not the 'BPS' conditions. See further for the incorporation of those.

This is the case for the [42] and [321] irreps in all classes, as well as for the [6]. Whereas this last fact is obvious (it corresponds to the abelian case), the general reason for the other ones is unclear.

- If the values of the coefficients are fixed numbers for some, and arbitrary parameters for other combinations, we have separated the basis accordingly. This is the case for all the $[2^3]$ irreps, where for each class a single combination is fixed, and for the $[31^3]$ likewise.

- The stand–alone $[21^4]$ invariant (which has arbitrary coefficient as well) is not touched.

In table 4, we have again listed the potential cyclic invariants in the group-theoretic classification, in the changed basis. The resulting sixth order terms in the action are

$$
\begin{aligned}
\mathcal{L}^{(6)} \;=\; & \frac{1}{720}I_6^6 + \frac{1}{6480}I_{42}^6 - \frac{1}{5760}I_{321}^6 + \frac{1}{720}I_{222}^6 \\
& +\lambda_1\, I_{222}^{'6} + \lambda_2\, I_{222}^{''6} + \lambda_3\, I_{222}^{'''6} + \lambda_4\, I_{3111}^{'6} + \lambda_5\, I_{21111}^{'6} \\
& -\frac{1}{480}I_6^{42} + \frac{1}{3240}I_{42}^{42} - \frac{1}{11520}I_{321}^{42} - \frac{1}{360}I_{222}^{42} + \lambda_6\, I_{222}^{'42} \\
& +\frac{1}{5760}I_6^{222} - \frac{1}{25920}I_{42}^{222} - \frac{1}{2880}I_{222}^{222} + \lambda_7\, I_{222}^{'222}\,.
\end{aligned}
\tag{31}
$$

It is clear that the resulting expression displays a remarkable amount of structure, but we have not been able to penetrate beyond the obvious.

An important check on the arbitrariness is provided by the fact that some commutator combinations can not possibly contribute in the restricted class of background that we investigated. These are, in an obvious notation (see appendix D if an explanation is needed)

$$
Tr[F_1 F_{1'}][F_2 F_{2'}][F_3 F_{3'}]\,, \tag{32}
$$
$$
Tr[F_1 F_{1'}][F_2 F_{2'}][F_3 F_4]\,, \tag{33}
$$
$$
Tr[F_1 F_2][F_3 F_4][F_5 F_6]\,, \tag{34}
$$
$$
Tr[F_1 F_2][F_3 F_5][F_4 F_6]\,, \tag{35}
$$
$$
Tr[F_1 F_2][F_2 F_5][F_4 F_6]\,. \tag{36}
$$

The reason is obvious: the quadratic variation of these products of three commutators always has one commutator left, and vanishes since the

Class	S_6-rep	Invariant linear combination	coefficient	name
2 2 2	[6]	$3i_1 + 2i_2 + 1i_3 + 3i_4 + 6i_5$	abel	I_6^{222}
2 2 2	[42]	$1i_1 + 4i_2 - 3i_3 - 4i_4 + 2i_5$	fixed	I_{42}^{222}
2 2 2	[42]	$2i_1 + 1i_2 + 1i_3 - 1i_4 - 3i_5$	0	
2 2 2	[2^3]	$-3i_1 + 2i_2 + 1i_3 + 0i_4 + 0i_5$	fixed	I_{222}^{222}
2 2 2	[2^3]	$0i_1 - 1i_2 + 1i_3 - 3i_4 + 3i_5$	undet	$I_{222}'^{222}$
4 2	[6]	$2, 2, 2, 2, 2, 2, 1, 1, 1$	abel	I_6^{42}
4 2	[42]	$2, 0, 2, 1, -1, -1, 0, -2, -1$	fixed	I_{42}^{42}
4 2	[42]	$-1, 3, 2, -2, -1, -1, 0, 1, -1$	0	
4 2	[42]	$3, 3, -2, -2, 3, 3, -6, -1, -1$	0	
4 2	[42]	$-6, 0, 1, 4, 3, 3, 0, -1, -4$	0	
4 2	[321]	$-2, 0, 2, -2, 1, 1, 0, -2, 2$	fixed	I_{321}^{42}
4 2	[321]	$0, 0, 0, 0, 1, -1, 0, 0, 0$	0	
4 2	[2^3]	$0, 2, -2, 0, 0, 0, 1, -1, 0$	fixed	I_{222}^{42}
4 2	[2^3]	$1, -1, 0, -2, 1, 1, 1, 0, -1$	undet	$I_{222}'^{42}$
6	[6]	$1, 6, 6, 3, 6, 6, 6, 6, 2, 3, 6, 3, 3, 3$	abel	I_6^6
6	[42]	$1, 2, -2, -1, -2, -2, -6, -2, -2, 3, 2, 5, 3, 1$	fixed	I_{42}^6
6	[42]	$1, 2, 1, 2, 1, 1, -3, -2, 1, 0, -1, -1, 0, -2$	0	
6	[42]	$1, 6, 1, -2, 1, 1, 1, 6, -3, -2, -9, 3, -2, -2$	0	
6	[42]	$2, -3, -4, 5, -4, -4, 2, 3, 3, -1, -3, 3, -1, 2$	0	
6	[321]	$2, 2, 1, 0, 1, -4, 2, -2, -2, -1, 2, -2, -1, 2$	fixed	I_{321}^6
6	[321]	$0, 0, -1, 0, 1, 0, 0, 0, 0, 1, 0, 0, -1, 0$	0	
6	[2^3]	$1, 2, -2, -3, -2, 2, 2, -2, 2, -1, 2, 1, -1, -1$	fixed	I_{222}^6
6	[2^3]	$1, -4, 1, 0, 1, 5, -1, -2, -1, -1, -1, 1, -1, 2$	undet	$I_{222}'^6$
6	[2^3]	$0, 2, -2, 0, -2, 2, 2, -2, 0, 2, -4, -2, 2, 2$	undet	$I_{222}''^6$
6	[2^3]	$0, -2, 2, 0, 2, -2, 4, -4, 0, 1, -2, 2, 1, -2$	undet	$I_{222}'''^6$
6	[31^3]	$0, 0, -2, 0, 2, 0, 0, 0, 0, -1, 0, 0, 1, 0$	fixed=0	I_{3111}^6
6	[31^3]	$1, -2, -1, 0, -1, 1, 1, 2, -1, 1, 1, -1, 1, -2$	undet	$I_{3111}'^6$
6	[21^4]	$1, -2, 2, -3, 2, -2, -2, 2, 2, 1, -2, -1, 1, 1$	undet	$I_{21111}'^6$

Table 4: Results on the coefficients of cyclic invariants by irreducible representation. The last column contains names for future reference. The column before last has the following meaning: 'abel' indicates a coefficient fixed by the abelian case (or by Tseytlin's proof of the 'symmetric trace' formula), 'fixed' and 'undet' mean fixed resp. undetermined by the rescaled Yang–Mills analysis.

background is abelian.[15] The first line corresponds to λ_7, the second to λ_6. The last three lines generate through linear combinations the [3111]–invariant with λ_4, as well as the [222] invariants with λ_1 and λ_2.

6   BPS configurations

Turning on magnetic fields usually breaks all supersymmetry with as a result that the D-brane configuration becomes unstable. This can already be seen from the massformulae, eqs. (5) and (7), which exhibit the generic presence of tachyonic modes in the spectrum. However, it was noticed in [11] that for very specific choices of the background some supersymmetry survives.

We will first formulate this in the T-dual picture. We take two Dp-branes, one of them in the $(2, 4, \cdots, 2p)$ direction and the other one rotated over an angle γ_1 in the $(2\,3)$ plane, over an angle γ_2 in the $(4\,5)$ plane, ..., over an angle γ_p in the $(2p\,2p+1)$ plane. Searching for common directions in the supersymmetry charge and the rotated charge gives BPS configurations which are summarized below.

p	BPS angle	susy's	BPS magnetic fields
2	$\gamma_1 = \gamma_2$	8	$f_1^3 = f_2^3$
3	$\gamma_1 = \gamma_2 + \gamma_3$	4	$f_1^3 = f_2^3 + f_3^3 + f_1^3 f_2^3 f_3^3$
4	$\gamma_1 = \gamma_2 + \gamma_3 + \gamma_4$	2	$f_1^3 = f_2^3 + f_3^3 + f_4^3 + f_1^3 f_2^3 f_3^3 +$ $f_1^3 f_3^3 f_4^3 + f_1^3 f_2^3 f_4^3 - f_2^3 f_3^3 f_4^3$
	$\gamma_1 = \gamma_2,\ \gamma_3 = \gamma_4$	4	$f_1^3 = f_2^3,\ f_3^3 = f_4^3$
	$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$	6	$f_1^3 = f_2^3 = f_3^3 = f_4^3$

We assumed that none of the angles are zero. In the table we list the conditions on the angles, the number of preserved supercharges and finally the T-dual picture where the condition on the angles is translated, using eqs. (6) and (4), into a condition on the magnetic fields. For simplicity we took the magnetic field entirely in the σ_3 direction, i.e. f_i^0 in equation (8) vanishes.

Though the conditions on the angles are linear, they translate for two cases into non-linear conditions on the magnetic fields. In fact, when

---

[15]It is less obvious that the (independent) combination $F_1 F_{[2} F_{|3|} F_4 F_5 F_{6]}$ (leaving the label 3 out of the antisymmetrisation) also does not contribute: here the block-diagonal nature of the background is involved. This is in fact the invariant $I'^{\,6}_{21111}$ with coefficient λ_5. We have no short explanation for the seventh invariant, with λ_3.

switching on the $U(1)$ part of the magnetic field, one always gets such corrections. At first sight one would expect this to give a crucial handle on the NBI. Indeed, BPS configurations should solve the equations of motion with as a result that the non-linear conditions relate different orders in the NBI. However all backgrounds considered above are in the torus of $U(2)$ and thus insensitive to different orderings in the equations of motion. In fact they all solve the equations of motion of the abelian Born-Infeld action and as a consequence those arising from our action through order F^6 as well. There is one case where we do have a good guess for the general BPS condition: rotated D2-branes or D4-branes with magnetic fields. In that case the obvious guess for the full non-abelian BPS condition is self-duality of the magnetic field.

In [12], some arguments were put forward to sustain the claim that self-dual static magnetic backgrounds solving the equations of motion while simultanously minimizing the energy is equivalent to demanding that the whole NBI for such configurations collapses to the leading Yang-Mills term. It was shown that the symmetrized trace prescription does share this property. Implementing this assumption in our case gives five conditions on the general form of the action at sixth order. It turns out that three of these are dependent on the previously implemented rescaled Yang–Mills conditions, thus providing a consistency check on both our results and the proposal in [12]. The remaining two take an extremely simple form in terms of the coefficients in eq. (31), viz.

$$
\begin{aligned}
\lambda_3 &= \frac{1}{1440}\,, \\
\lambda_7 &= \lambda_1 - \frac{\lambda_2}{4} + \frac{\lambda_6}{2}\,.
\end{aligned}
$$

Note that different Lorentz contraction structures are connected[16]. As far as the permutation group structure is concerned, the conditions are pure [222].

What about the full non-abelian version of the non-linear BPS conditions? While the first order correction to the linear relations can easily be deduced from the fact that they should solve the equations of motion through order F^3, nothing can be said to all orders yet. A more detailed study of these BPS configurations and their consequences for the NBI, has to wait for a better understanding of supersymmetry in the NBI [13].

---

[16] Selfduality, $F_{\mu\nu} = \varepsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}/2$, implies that $F_{\mu\rho}F^\rho{}_\nu + F_{\nu\rho}F^\rho{}_\mu = \eta_{\mu\nu}F_{\rho\sigma}F^{\sigma\rho}/2$. Repeatedly using this, allows us to rewrite all terms of class 6 and 4 2 in terms of the five elements of the class 2 2 2.

7 Conclusions

In this paper, we made a first systematic attempt to determine corrections to the Str-terms in the NBI action. The physical testing ground on which we worked, were D-branes wrapped on tori with magnetic backgrounds turned on. Central in an important part of our analysis is the fact that the spectrum for open strings stretching between these D-branes as predicted by perturbative string theory differs by a mere rescaling from the YM-approximation to the spectrum. We made two bold assumptions to proceed in unknown territory. The first was that we did not take along *all* possible derivative corrections to the NBI. This was inspired partly by practical motives, partly by the second assumption, namely, that the action quadratic in the fluctuations in this background obtained from the NBI should be a rescaled YM action. Under these assumptions we were able to put severe constraints on the NBI action. A weak a posteriori argument is that this method yields correct results at order 4. More encouraging is the fact that this approach yields constraints that are compatible with the constraints we obtained from analysing BPS configurations – this was not evident from the outset.

By construction, this action heals a severe default of the Str NBI action, pointed out in [4] [9] and [5], namely that it doesn't predict the correct spectrum for open strings on our testing ground. A direct calculation of the terms at order 6 of the NBI via six point functions in string theory or a five loop beta-function calculation in a non-linear $\sigma-$ model would be welcome, of course, to see whether our assumptions are valid. As long as this calculation is not available, other methods to get a grip on the NBI are worth study. A natural extension of our ideas is to just enlarge the testing ground by looking at other compactification manifolds, and by looking at different, possibly electric backgrounds. It could help also if we could gain more insight into the relative norms of the linear combinations of diagrams corresponding to double cosets, for instance, to see whether there might be a systematic expansion for the NBI – although this is perhaps asking for too much [17].

There are of course alternative techniques that are more complementary to our approach. The most promising route to obtain a grip on the non-abelian Born-Infeld might be via supersymmetry. Simply by noethering one can try to modify ten dimensional YM with non-linear corrections and modify the supersymmetry transformation rules accordingly. In the abelian case this fixes e.g. uniquely the fourth order term

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[17]We thank J. de Boer for a discussion on this point.

in the BI action [14]. Because of the severely restricted form of the supersymmetry algebra in ten dimensions, it might even be[18] that the BI action is the only supersymmetric deformation of abelian YM. Continuing this line of thought, it should be clear that a similar analysis should be performed in the non-abelian case. A good starting point would be the BPS conditions in higher dimensions in section 6 – they provide a hint of how to modify the supersymmetry variations. It seems important to us to carry out this program in the maximum of ten dimensions, because the supersymmetry algebra is largest there and therefore puts (much) stronger restrictions on the form of the action. For lower dimensions some partial results for a supersymmetric non-abelian extension of Born-Infeld theory are available [15].

Closely linked to the idea of a supersymmetric action on the brane is the idea of the construction of an action invariant under κ–symmetry [16]. This approach starts from the observation that the form of the Wess-Zumino term, which describes the coupling of the gauge fields to the Ramond-Ramons bulkfields, is severely restricted, even in the non-abelian case. As the variation of the Wess-Zumino term under the κ-transformations has to be cancelled by the variation of the NBI, one gets a recursive method to construct the NBI. This program was already carried out through quartic order in the Yang-Mills field strength, and including all fermion bilinear terms up to terms cubic in the field strength [16]. The orderings are indeed completely fixed by requiring κ-invariance. Surprisingly, it was found that at such low order, deviations of the symmetrized trace proposal do already appear.

Another route to the NBI derives from the study of the equivalence of non-commutative and commutative Born-Infeld actions via the Seiberg-Witten map. In this way one obtains constraints on derivative corrections to the Born-Infeld action. It would be interesting to see whether this can teach us anything about the NBI action, as claimed in [17].

Solutions to different non-abelian extensions of Born-Infeld theory, not necessarily related to string theory have been studied. The non-linearity of the action often leads to a smoothing out of solutions to an ordinary YM or Maxwell action. It could be interesting to study what kind of corrections can be expected from more general proposals for non-abelian extensions of Born-Infeld theory, including ours.

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[18]This was suggested to us by Savdeep Sethi.

Acknowledgements

A Drawing diagrams

We draw here all the diagrams corresponding to the double cosets as introduced in section 4. [19]
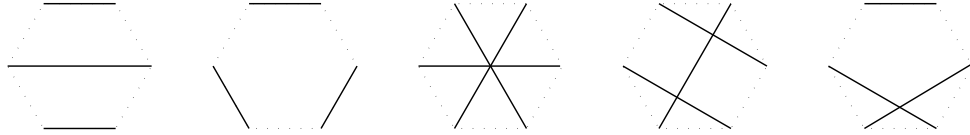

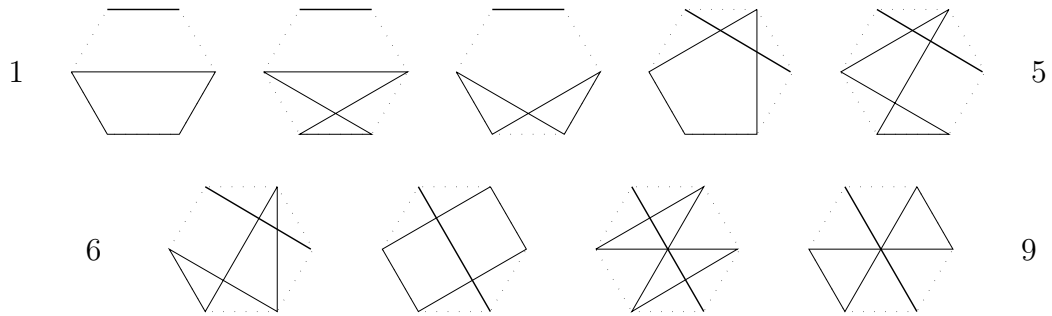Figure 2: Diagrammatic representation of the five [222] terms in the action.

Figure 3: Diagrammatic representation of the nine [42] terms in the action, numbered as indicated, from left to right.

---

[19]In fact, it was intuitively easy to see, even before we knew the group theory from section 4 that drawing all different diagrams was sufficient to enumerate all different terms in the action.
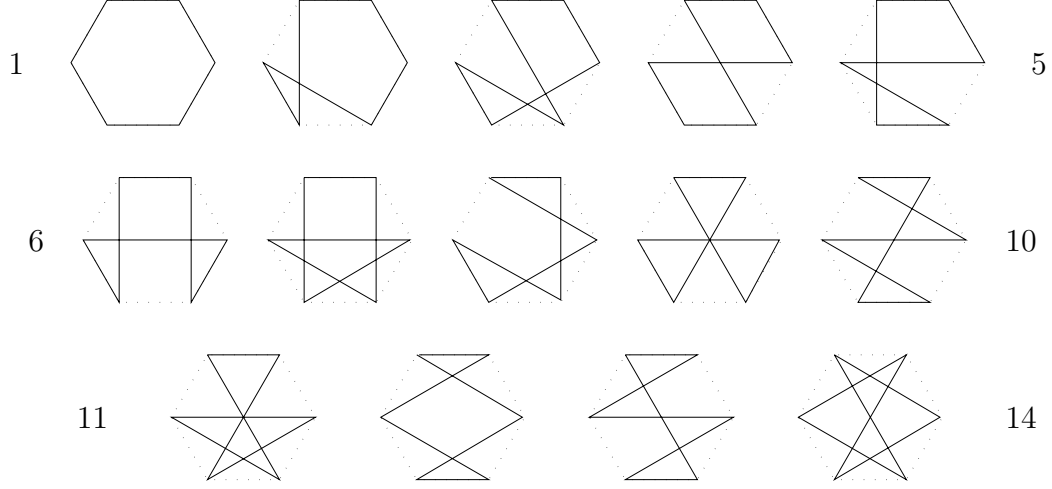
23

Figure 4: Diagrammatic representation of the fourteen [6] terms in the action.

In (37) we give a few examples in our ansatz for the action that should leave no ambiguity as to which terms in the action the diagrams correspond to. The upper index on a_m^{nnn} indicates the class, the lower index the number of the diagram:

$$
\begin{aligned}
\mathcal{L} = Tr(&\ldots \\
+&\ \ldots + a_3^6 F_{\alpha_1\alpha_2} F_{\alpha_2\alpha_3} F_{\alpha_3\alpha_4} F_{\alpha_6\alpha_1} F_{\alpha_4\alpha_5} F_{\alpha_5\alpha_6} \\
+&\ \ldots + a_4^6 F_{\alpha_1\alpha_2} F_{\alpha_2\alpha_3} F_{\alpha_3\alpha_4} F_{\alpha_6\alpha_1} F_{\alpha_5\alpha_6} F_{\alpha_4\alpha_5} \\
+&\ \ldots + a_{14}^6 F_{\alpha_1\alpha_2} F_{\alpha_4\alpha_5} F_{\alpha_2\alpha_3} F_{\alpha_6\alpha_1} F_{\alpha_3\alpha_4} F_{\alpha_5\alpha_6} \\
+&\ \cdots + a_1^{2,4} F_{\alpha_1\alpha_2} F_{\alpha_2\alpha_1} F_{\beta_1\beta_2} F_{\beta_2\beta_3} F_{\beta_3\beta_4} F_{\beta_4\beta_1} \\
+&\ \cdots + a_3^{2,2,2} F_{\alpha_1\alpha_2} F_{\beta_1\beta_2} F_{\gamma_1\gamma_2} F_{\alpha_2\alpha_1} F_{\beta_2\beta_1} F_{\gamma_2\gamma_1} \\
+&\ O(F^8) \,. \hspace{4cm} (37)
\end{aligned}
$$

B   Table

The table in this appendix records the result of the construction of a basis of invariants based on permutation group analysis. The notation is as follows.

The first five lines correspond to the class of terms with three times a double contraction of Lorentz indices, labeled class 222. The second

Class	S_6-rep	Prefactor	Invariant linear combination
2 2 2	[6]	$\frac{1}{15}$	$3i_1 + 2i_2 + i_3 + 3i_4 + 6i_5$
2 2 2	[42]	$\frac{1}{10}$	$2i_1 + 3i_2 - i_3 - 3i_4 - i_5$
2 2 2	[42]	$\frac{1}{20}$	$-3i_1 - 2i_2 - i_3 + 2i_4 + 4i_5$
2 2 2	$[2^3]$	$\frac{1}{12}$	$-6i_1 + 5i_2 + i_3 + 3i_4 - 3i_5$
2 2 2	$[2^3]$	$\frac{1}{12}$	$0i_1 - i_2 + i_3 - 3i_4 + 3i_5$
4 2	[6]	$\frac{1}{15}$	$2, 2, 2, 2, 2, 2, 1, 1, 1$
4 2	[42]	$\frac{1}{10}$	$3, -3, 0, 2, -1, -1, 1, -2, 1$
4 2	[42]	$\frac{1}{20}$	$5, 3, 4, -2, -3, -3, -1, -2, -1$
4 2	[42]	$\frac{1}{20}$	$-4, 2, 2, 0, 0, 0, 1, 1, -2$
4 2	[42]	$\frac{1}{20}$	$-4, -2, -2, 4, 4, 4, -1, -1, -2$
4 2	[321]	$\frac{1}{15}$	$-1, 0, 1, -1, -2, 3, 0, -1, 1$
4 2	[321]	$\frac{1}{15}$	$-1, 0, 1, -1, 3, -2, 0, -1, 1$
4 2	$[2^3]$	$\frac{1}{12}$	$1, -5, 4, -2, 1, 1, -1, 2, -1$
4 2	$[2^3]$	$\frac{1}{6}$	$1, 1, -2, -2, 1, 1, 2, -1, -1$
6	[6]	$\frac{1}{60}$	$1, 6, 6, 3, 6, 6, 6, 6, 2, 3, 6, 3, 3, 3$
6	[42]	$\frac{1}{20}$	$1, -1, -1, 3, -1, -1, 0, 0, 2, -1/2, -1, 0, -1/2, 0$
6	[42]	$\frac{1}{40}$	$1, 4, 1, 0, 1, 1, -5, -2, -1, 1, -1, 1, 1, -2$
6	[42]	$\frac{1}{20}$	$1, 2, -1, 0, -1, -1, -1, 2, -1, 0, -3, 3, 0, 0$
6	[42]	$\frac{1}{20}$	$1, 2, 1, 2, 1, 1, -1, 0, 1, -1, -3, -1, -1, -2$
6	[321]	$\frac{1}{45}$	$4, 4, 2, 0, 2, -8, 4, -4, -4, -2, 4, -4, -2, 4$
6	[321]	$\frac{1}{45}$	$1, 1, 3, 0, -2, -2, 1, -1, -1, -3, 1, -1, 2, 1$
6	$[2^3]$	$\frac{1}{12}$	$1, -4, 2, -1, 2, 2, 2, -4, 0, -1, 0, 3, -1, -1$
6	$[2^3]$	$\frac{1}{36}$	$1, -2, -1, 0, -1, 7, 1, -4, -1, 1, -5, -1, 1, 4$
6	$[2^3]1$	$\frac{1}{12}$	$1, 3, -3, -3, -3, 3, 3, -3, 2, 0, 0, 0, 0, 0$
6	$[2^3]$	$\frac{1}{24}$	$1, 2, -3, -2, -3, 5, -1, 0, 1, -1, 1, -1, -1, 2$
6	$[31^3]$	$\frac{1}{18}$	$0, 0, -1, 0, 1, 0, 0, 0, 0, -1/2, 0, 0, 1/2, 0$
6	$[31^3]$	$\frac{1}{18}$	$1, -2, -3, 0, 1, 1, 1, 2, -1, 0, 1, -1, 2, -2$
6	$[21^4]$	$\frac{1}{36}$	$1, -2, 2, -3, 2, -2, -2, 2, 2, 1, -2, -1, 1, 1$

Table 5: Cyclic invariants by irreducible representation

column gives the permutation group class in the standard cycle notation, and the third gives the corresponding invariant. The combination is written as a weighted sum of diagrams i_n, the latter labeled in the order given in figure 2 (see appendix A). In the following lines, this information is given for the classes of terms with Lorentz contractions following the patterns 42 and 6 respectively. For the invariant linear combinations we just give the coefficients, again corresponding to the figures (3 and 4) in the preceding appendix.

C Abelian constraint

We know the Born-Infeld action for the gauge group $U(1)$. After expanding the determinant and the square root it looks as follows:

$$
\begin{aligned}
\mathcal{L}^6 \;=\;\; & \frac{1}{4} F^{\alpha\beta} F_{\beta\alpha} \\
& + \frac{1}{8} F^{\alpha\beta} F_{\beta\gamma} F^{\gamma\delta} F_{\delta\alpha} - \frac{1}{32} (F^{\alpha\beta} F_{\alpha\beta})^2 \\
& + \frac{1}{12} F^{\alpha\beta} F_{\beta\gamma} F^{\gamma\delta} F_{\epsilon\zeta} F^{\zeta\iota} F_{\iota\alpha} - \frac{1}{32} F^{\alpha\beta} F_{\beta\alpha} F^{\gamma\delta} F_{\delta\zeta} F^{\zeta\iota} F_{\iota\gamma} \\
& + \frac{1}{384} (F^{\alpha\beta} F_{\beta\alpha})^3 \\
& + O(F^8) \,.
\end{aligned}
\tag{38}
$$

From this we derive the following constraints on the coefficients in our general ansatz:

$$
\begin{aligned}
a_1^2 &= \frac{1}{4} \,, \\
a_1^4 + a_2^4 &= \frac{1}{8} \,, \\
a_1^{2,2} + a_2^{2,2} &= -\frac{1}{32} \,,
\end{aligned}
\tag{39}
$$

up to fourth order, and

$$
\begin{aligned}
\sum_{i=1}^{14} a_i^6 &= \frac{1}{12} \,, \\
\sum_{i=1}^{9} a_i^{2,4} &= -\frac{1}{32} \,, \\
\sum_{i=1}^{5} a_i^{2,2,2} &= \frac{1}{384} \,,
\end{aligned}
\tag{40}
$$

at sixth order.

D Some Technical details

Action quadratic in fluctuations in a magnetic background

We split the field strength in background and fluctuations, $F = \mathcal{F} + \delta F$, and the fluctuations into a part linear in the gauge field fluctuations and a part quadratic in the gauge field fluctuations:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta]\,, \tag{41}$$

$$\delta F_{\alpha\beta} = \delta_1 F_{\alpha\beta} + \delta_2 F_{\alpha\beta}\,, \tag{42}$$

$$\delta_1 F_{\alpha\beta} = D_\alpha \delta A_\beta - D_\beta \delta A_\alpha\,, \tag{43}$$

$$\delta_2 F_{\alpha\beta} = i[\delta A_\alpha, \delta A_\beta]\,, \tag{44}$$

$$D_\alpha = \partial_\alpha + i[A_\alpha, .]\,. \tag{45}$$

We substitute $F = \mathcal{F} + \delta_1 F + \delta_2 F$ into the action up to order F^6 and restrict to the terms quadratic in the fluctuations. We'll get terms proportional to $\delta_2 F$ and terms proportional to $(\delta_1 F)^2$.

Terms proportional to $\delta_2 F$

When we choose $\mathcal{F}$ in the CSA we can write the terms proportional to $\delta_2 F$ as:

$$
\begin{aligned}
\mathcal{L}_2^{(2)} &= \frac{1}{2}\delta_2 F^{\alpha\beta}\mathcal{F}_{\beta\alpha} \\
&+ \frac{1}{2}\delta_2 F^{\alpha\beta}\mathcal{F}_{\beta\gamma}\mathcal{F}^{\gamma\delta}\mathcal{F}_{\delta\alpha} - \frac{1}{8}\delta_2 F^{\alpha\beta}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\delta\gamma}\mathcal{F}_{\delta\gamma} \\
&+ \frac{1}{2}\delta_2 F^{\alpha\beta}\mathcal{F}_{\beta\gamma}\mathcal{F}^{\gamma\delta}\mathcal{F}_{\delta\zeta}\mathcal{F}^{\zeta\iota}\mathcal{F}_{\iota\alpha} - \frac{1}{16}\delta_2 F^{\alpha\beta}\mathcal{F}_{\beta\alpha}\mathcal{F}^{\gamma\delta}\mathcal{F}_{\delta\zeta}\mathcal{F}^{\zeta\iota}\mathcal{F}_{\iota\gamma} \\
&- \frac{1}{8}\delta_2 F^{\gamma\delta}\mathcal{F}_{\delta\zeta}\mathcal{F}^{\zeta\iota}\mathcal{F}_{\iota\gamma}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta\alpha} + \frac{1}{64}\delta_2 F_{\alpha\beta}\mathcal{F}^{\beta\alpha}(\mathcal{F}^{\gamma\delta}\mathcal{F}_{\gamma\delta})^2\,.
\end{aligned}
\tag{46}
$$

This part of the action naturally has the same coefficients as the abelian action.

Terms proportional to $(\delta_1 F)^2$

The $U(2)$ components of $\delta_1 F$ (see eq. (43) are denoted as in $\delta_1 F = \sum_{n=1,2}\delta_1 F^{(n)}\sigma_n$ and the background splits likewise as $\mathcal{F} = \mathcal{F}^0 + \mathcal{F}^3 \sigma_3$.

For the Lorentz index contraction we use a shorthand notation indicating the sequence(s) of contractions, easily understood and generalised from the following hypothetical example:

$$A_{\mu_1\mu_2}B_{\mu_3\mu_1}C_{\nu_1\nu_2}D_{\mu_2\mu_3}E_{\nu_2\nu_1} \longrightarrow A_1 B_3 C_{1'} D_2 E_{2'}.$$

Our calculation to order four gives for the off-diagonal fluctuations:

$$
\begin{aligned}
\mathcal{L}_1^{(2)} = \quad & 2a_1^2(\delta_1 F_1^1 \delta_1 F_2^1 + \delta_1 F_1^2 \delta_1 F_2^2) \\
+ \quad & 2(\delta_1 F_1^1 \delta_1 F_2^1 + \delta_1 F_1^2 \delta_1 F_2^2)[(4a_1^4 + 4a_2^4)\mathcal{F}_3^0 \mathcal{F}_4^0 + 4a_1^4 \mathcal{F}_3^3 \mathcal{F}_4^3] \\
+ \quad & 2(\delta_1 F_1^1 \delta_1 F_3^1 + \delta_1 F_1^2 \delta_1 F_3^2)[(2a_1^4 + 2a_2^4)\mathcal{F}_2^0 \mathcal{F}_4^0 + (-2a_1^4 + 2a_2^4)\mathcal{F}_2^3 \mathcal{F}_4^3] \\
+ \quad & 2(\delta_1 F_1^1 \delta_1 F_2^1 + \delta_1 F_1^2 \delta_1 F_2^2) \times \\
& \quad [(2a_1^{2,2} + 2a_2^{2,2})\mathcal{F}_{1'}^0 \mathcal{F}_{2'}^0 + (2a_1^{2,2} - 2a_2^{2,2})\mathcal{F}_{1'}^3 \mathcal{F}_{2'}^3] \quad\quad\quad (47) \\
+ \quad & 2(\delta_1 F_1^1 \delta_1 F_{1'}^1 + \delta_1 F_1^2 \delta_1 F_{1'}^2)[(4a_1^{2,2} + 4a_2^{2,2})\mathcal{F}_2^0 \mathcal{F}_{2'}^0 + (4a_2^{2,2})\mathcal{F}_2^3 \mathcal{F}_{2'}^3] \; .
\end{aligned}
$$

The corresponding expressions for the general form (with our restrictions) of the action at order F^6 are not very illuminating, and we refrain from giving them explicitely.

Background blockdiagonal in Lorentz indices

After filling in the background we obtain, from the quadratic terms:

$$
\begin{aligned}
\mathcal{L}^{(2)} = \quad & (\delta_1 F_{0,2i-1}^{(a)})^2 + (\delta_1 F_{0,2i}^{(a)})^2 \\
& - (\delta_1 F_{2i-1,2i}^{(a)})^2 \\
& - \frac{1}{2}\sum_{i\neq j}((\delta_1 F_{2i-1,2j-1}^{(a)})^2 + (\delta_1 F_{2i-1,2j}^{(a)})^2 + (\delta_1 F_{2i,2j-1}^{(a)})^2 + (\delta_1 F_{2i,2j}^{(a)})^2) \\
& - 2\delta_2 F_{2i-1,2i}^{(3)} f_i^3, \quad\quad\quad (48)
\end{aligned}
$$

and from the quartic terms we find:

$$
\begin{aligned}
\mathcal{L}^{(4)} = \quad & -8[(\delta_1 F_{0,2i-1}^{(a)})^2 + (\delta_1 F_{0,2i}^{(a)})^2][(a_1^4 + a_2^4)(f_i^0)^2 + a_1^4(f_i^3)^2 \\
& \quad\quad\quad + (2a_1^{2,2} + 2a_2^{2,2})(f_k^0)^2 + (2a_1^{2,2} - 2a_2^{2,2})(f_k^3)^2] \\
& + 8(\delta_1 F_{2i-1,2i}^{(a)})^2[(3a_1^4 + 3a_2^4 + 4a_1^{2,2} + 4a_2^{2,2})(f_i^0)^2 \\
& \quad\quad\quad\quad + (a_1^4 + a_2^4 + 4a_2^{2,2})(f_i^3)^2 \\
& \quad\quad\quad\quad + (2a_1^4 + 2a_2^4)(f_k^0)^2 + (2a_1^{2,2} - 2a_2^{2,2})(f_k^3)^2] \\
& + 8\sum_{i\neq j}[(\delta_1 F_{2i-1,2j-1}^{(a)})^2 + (\delta_1 F_{2i-1,2j}^{(a)})^2 + (\delta_1 F_{2i,2j-1}^{(a)})^2 + (\delta_1 F_{2i,2j}^{(a)})^2] \times
\end{aligned}
$$

$$\times[(a_1^4 + a_2^4)(f_i^0)^2 + (a_1^4)(f_i^3)^2 + (a_1^{2,2} + a_2^{2,2})(f_k^0)^2 + (a_1^{2,2} - a_2^{2,2})(f_k^3)^2]$$
$$+8(\delta_1 F_{2i-1,2i}^{(a)} \delta_1 F_{2j-1,2j}^{(a)})[(a_1^4 + a_2^4 + 4a_1^{2,2} + 4a_2^{2,2})f_i^0 f_j^0$$
$$+ (-a_1^4 + a_2^4 + 4a_2^{2,2})f_i^3 f_j^3]$$
$$+\delta_2 F_{2i-1,2i}^{(3)} f_i^3 [6(f_i^0)^2 + 2(f_i^3)^2 - (f_k^0)^2 - (f_k^3)^2]$$
$$+\delta_2 F_{2i-1,2i}^{(3)} f_i^0 (-2f_k^0 f_k^3)$$
$$+8 \sum_{i \neq j} \delta_2 F_{2i-1,2i}^{(3)} f_i^3 [(a_1^4 + a_2^4)f_i^0 f_j^0 + (-a_1^4 + a_2^4)f_i^3 f_j^3] . \tag{49}$$

The sixth order calculation is analogous, the results are omitted.

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